A Thermodynamic Approach to the Analysis of Multi-Robot Cooperative Localization Under Independent Errors.

Yotam Elor and Alfred M. Bruckstein.
Computer Science Department.
Technion, Haifa 32000, Israel.

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Abstract

We propose a new approach to the simultaneous cooperative localization of a group of robots capable of sensing their own motion and the relative position of nearby robots. In the last decade, the use of distributed optimal Kalman filters (KF) to solve this problem have been studied extensively. In this paper, we propose to use a sub-optimal Kalman filter (denoted by EA). EA requires significantly less computation and communication resources then KF. Furthermore, in some cases, EA provides better localization.

In this paper EA is analyzed in a soft "thermodynamic" fashion i.e. relaxing assumptions are used during the analysis. The goal is not to derive hard lower or upper bounds but rather to characterize the robots expected behavior. In particular, to predict the expected localization error. The predictions were validated using simulations. We believe that this kind of analysis can be beneficial in many other cases.

1 Introduction

Localization is the task of estimating the robot location. It has been identified as one of the key problems in robotics. The localization problem can be roughly divided into two variants: In the first variant the robots can estimate their location by sensing their surroundings and comparing it with information they posses about the environment (e.g. a map). While in the second variant, the robots have no such capabilities. Instead, every robot knows its initial location and updates its location estimation based on odometry readings.

There is a vast body of literature discussing the first variant where the main challenge is to incorporate the large quantity of data gathered by the robot (or robots) into a consisted world view. The means to gather data are the *exteroceptive* sensors which surveys the world and the *proprioceptive* sensors (odometry) which monitor the robot itself e.g. GPS, compass, wheel encoders, etc. This paper's focus is on the second variant, the reader interested in the more general scenario is referred to the book by Borenstein *et al.*[1] and the survey of Thrun[2].

In the second variant of the localization problem it is assumed that, initially, every robot knows its location and uses odometry in order to track it (dead-reckoning). However, due to noisy sensor readings, in time, the estimation diverges from the robot real location. When a group of robots perform localization, the localization error can be reduced by sharing information between them. In order to do so, several *exteroceptive* capabilities are needed i.e. every robot is required to be able to sense the relative location of nearby robots and to communicate with them.

We denote the cooperative localization algorithm proposed in this is paper by "Error Averaging" (EA). In EA, every robot moves in the area while maintaining an estimate of its location using odometry. Whenever two robots are within sensing and communication range, they average their location estimations. In case the two robots' localization errors are uncorrelated, such an averaging will result in reducing the error of both robots[1].

[1]It is important to note that any odometry based approach will face difficulties when the odometry errors are

In this work, a large group of identical robots performing EA is considered. It is natural to apply thermodynamic approach when considering such a large and homogeneous group. This paper's goal is to characterize the behavior of the robots. In particular, we are interested in predicting the expected localization error. The goal is not to derive hard lower or upper bounds but rather to apply relaxing assumptions during the analysis in order to predict the expected behavior. The predictions are then validated using simulations.

A very simple "independent error" model (IEM) is considered. In IEM, the odometry errors are independent of the state of the robot. The localization error is accumulated as a two-dimension Gaussian. Furthermore, the errors added at different times are statistically independent. This model is relevant, for example, when considering a small flying platform in an indoor environment (no wind). In this case, the odometry errors are due to small air currents which are independent of the flying platform actions.

2 Preliminaries

The notations of [3] are adopted when possible. A group of M identical independent robots is considered. The robots move in a flat environment of size A. Every robot can communicate with its neighbors up to a constant limited distance V. The robots have sensors which enable them to detect nearby robots and sense their relative location (up to distance V). In order to keep the model simple, it is assumed that these sensors are error free i.e. the robots are able to sense the relative location of each other accurately. Extending our formulation to include noisy *exteroceptive* sensors is straightforward.

Two robots "meet" when the distance between them is at most V i.e. they can sense each other and communicate. Upon meeting, the robots average their location estimations hence reducing the localization error. So the frequency of meetings strongly effect the localization quality. Roughly speaking, the higher the frequency of collisions the lower the localization error. The frequency of collisions is determined by the movement pattern of the robots which in turn is application dependent. In this work it is assumed that the robots perform the following random walk: All robots travel at constant speed and the heading of every robot is generally fixed. However, upon hitting an obstacle, the robot randomize a new heading[2].

correlated. For example, consider a group of robots going uphill on a slippery terrain. The robots' wheels will tend to slip so the robots will measure forward speed higher then their actual speed. Since all robots' localization is biased toward the same direction, sharing information between the robots will not compensate for that error.

Discrete time is considered i.e. $t = 0, 1, 2....$ There are M robots modeled as points on the plane. The location of robot r_i in respect to a fixed reference frame is denoted by the vector $X_i(t) = [x_i(t), y_i(t), \phi_i(t)]^T$ where $x_i(t)$, $y_i(t)$ are the robot's coordinates and $\phi_i(t)$ is the robot's heading. Let v_0 be the (constant) robot speed and $\omega_i(t)$ its angular velocity at time t. The robot coordinates are updated in the normal way i.e.,

$$X_i(t+1) = X_i(t) + \begin{bmatrix} \cos(\phi_i(t)) & 0 \\ \sin(\phi_i(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ \omega_i(t) \end{bmatrix} \quad (1)$$

The location estimation of robot r_i at time t is denoted by $\hat{X}_i(t) = \left[\hat{x}_i(t), \hat{y}_i(t), \hat{\phi}_i(t)\right]^T$. Initially $\hat{X}(0) = X(0)$. The estimation error is given by the vector $\tilde{X} = \hat{X} - X = \left[\tilde{x}_i(t), \tilde{y}_i(t), \tilde{\phi}_i(t)\right]^T$.

$z \sim N(0, \sigma^2)$ implies that z is a random variable distributed normally in one-dimension with zero mean and variance of σ^2. For practical reasons we are interested in the expected error i.e. the expected distance between the robot location to where he thinks he is. The expected error for robot r_i is the expected value of $e_i = \sqrt{\tilde{x}_i^2(t) + \tilde{y}_i^2(t)}$ and is denoted by $E[e_i]$. In case $\tilde{x}_i \sim N(0, \sigma^2)$ and $\tilde{y}_i \sim N(0, \sigma^2)$, $E[e_i] = \sqrt{\frac{\pi}{2}\sigma^2}$.

In IEM, the localization errors added at each time step are independent of the robot state. It is assumed that the errors are distributed normally i.e.,

[2] In the experiments without obstacles, every robot randomize a new heading once in a long while in order to avoid static patterns.

$$\hat{X}_i(t+1) = \hat{X}_i(t) + \begin{bmatrix} \cos(\phi_i(t)) & 0 \\ \sin(\phi_i(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ \omega_i(t) \end{bmatrix} + \begin{bmatrix} N_{i,x}(t) \\ N_{i,y}(t) \\ 0 \end{bmatrix} \quad (2)$$

Where $N_{i,x}(t) \sim N(0,\sigma_0^2)$ $(N_{i,y}(t) \sim N(0,\sigma_0^2))$ is the noise added to $\hat{x}_i$ ($\hat{y}_i$) at time t and σ_0 is a constant. The errors are independent i.e. for any $i \neq j$ or $a \neq b$ (where $a, b \in \{x, y\}$) or $t_1 \neq t_2$: $N_{i,a}(t_1)$ and $N_{j,b}(t_2)$ are statistically independent.

Equation 2 can be written in the following manner

$$\hat{X}_i(t) = X_i(0) + \sum_{t'=0}^{t-1} \begin{bmatrix} N_{i,x}(t') \\ N_{i,y}(t') \\ 0 \end{bmatrix} \quad (3)$$
$$+ \sum_{t'=0}^{t-1} \begin{bmatrix} \cos(\phi_i(t')) & 0 \\ \sin(\phi_i(t')) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ \omega_i(t') \end{bmatrix}$$
$$= X_i(t) + \sum_{t'=0}^{t-1} \begin{bmatrix} N_{i,x}(t') \\ N_{i,y}(t') \\ 0 \end{bmatrix} \quad (4)$$

Hence,

$$\tilde{X}_i = \sum_{t'=0}^{t-1} \begin{bmatrix} N_{i,x}(t') \\ N_{i,y}(t') \\ 0 \end{bmatrix} \sim \begin{bmatrix} N(0, t \cdot \sigma_0^2) \\ N(0, t \cdot \sigma_0^2) \\ 0 \end{bmatrix} \quad (5)$$

So when no localization correction mechanisms are applied, the variance of the localization error grows linearly in time.

3 Average Free Path

In this section we will investigate the "average free path" i.e. the average distance traveled by a robot between meetings.

Fix a robot r. Let p_r be the probability of r meeting any other robot at time t. Denote by $r(t)$ the position of robot r at time t. Let V_{t-1} be the circle of radius V around $r(t-1)$ and V_t - the circle around $r(t)$, see Figure 1. So robot r meets robot r' at time t if and only if $r'(t) \in V_t$ and $r'(t-1) \notin V_{t-1}$. Note that in case $r'(t) \in V_t$ and $r'(t-1) \in V_{t-1}$ the distance between r and r' at time t is less than V. However, in this case, further exchange of information between the robots will not improve their localization (as shown in Section 4). So it is not considered as a meeting.

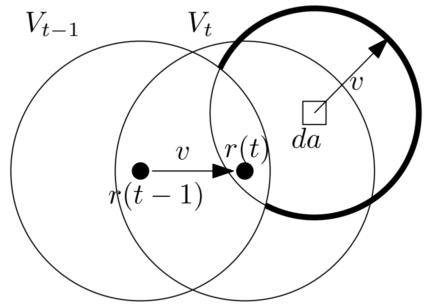


Figure 1

To derive the probability that there is a robot r' such as $r'(t) \in V_t$ and $r'(t-1) \notin V_{t-1}$, consider an infinitesimal area of size $da \subseteq V_t$ and a robot $r' \neq r$. Let

$$p_{r'} = Pr[r'(t) \in V_t \wedge r'(t-1) \notin V_{t-1}] \quad (6)$$
$$= \int_{V_t} Pr[r'(t) \in da \wedge r'(t-1) \notin V_{t-1}]\, da \quad (7)$$
$$= \int_{V_t} Pr[r'(t) \in da] \cdot Pr[r'(t-1) \notin V_{t-1} \,|\, r'(t) \in da]\, da \quad (8)$$

Assuming the robots are distributed uniformly over the area,

$$Pr[r'(t) \in da] = da/A \quad (9)$$

In case $r'(t) \in da$, it is assumed that $r'(t-1)$ is distributed uniformly over the circle of radius v_0 (the step length) with a center at $r'(t)$. So $Pr[r'(t-1) \notin V_{t-1} \,|\, r'(t) \in da]$ equals the part of this circle which is not inside V_0 (the bold arc in Figure 1). Let $\Delta x = x_{r'}(t) - x_r(t)$, $\Delta y = y_{r'}(t) - y_r(t)$. We face the problem of calculating the intersection of two circles at distance $d = \sqrt{(\Delta x + v_0)^2 + \Delta y^2}$, see Figure 2. θ is given

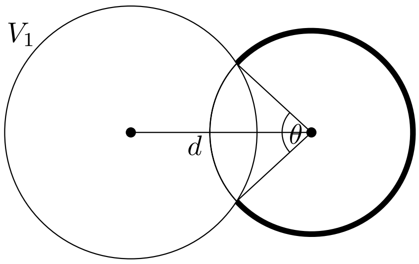


Figure 2

by (see [4]):

$$\theta = \begin{cases} 2\text{asin}\left(\frac{\sqrt{4d^2V^2-\left(d^2-v_0^2+V^2\right)^2}}{2dv_0}\right) & \text{if } r'(t) \notin V_{t-1} \\ 2\pi - 2\text{asin}\left(\frac{\sqrt{4d^2V^2-\left(d^2-v_0^2+V^2\right)^2}}{2dv_0}\right) & \text{if } r'(t) \in V_{t-1} \end{cases} \quad (10)$$

so

$$Pr\left[r'(t-1) \notin V_{t-1} \,|\, r'(t) \in da\right] = \frac{2\pi-\theta}{2\pi} \quad (11)$$

and finally

$$p_{r'} = \int\limits_{V_t} \frac{2\pi-\theta}{2\pi A} da \quad (12)$$

$p_{r'}$ can be calculated numerically. The probability that robot r meets any other robot at time t is given by

$$p_r = 1-(1-p_{r'})^{n-1} \quad (13)$$

Let δ be the time elapsed between two successive meetings of a specific robot with any other robot i.e. δ is the "free time" between meetings. δ is distributed geometrically with a mean of $1/p_r$, i.e.

$$Pr[\delta = k] = p_r(1-p_r)^{k-1} \quad (14)$$

p_r can not be measured experimentally. Instead, the distribution of δ was measured. The histogram of δ for three group sizes is presented in Figure 3. In those three runs the environment was a torus of size 100×100, $V = 3$ and $v_0 = 1$. The experiments show that the estimation of δ (hence p_r) is very accurate.

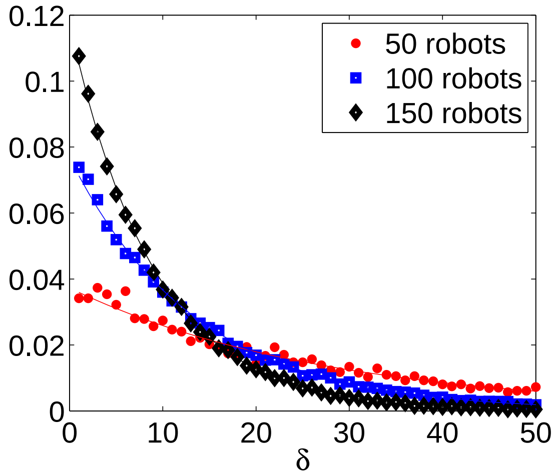


Figure 3: The histogram of δ. The solid lines are the theoretical estimations and the markers are the simulation results. Every line in the figure is a result of a single run of 5000 time steps.

4 The Covariance Matrix

In IEM, the components of X_i are independent so they can be analyzed separately. Hence only x component will be analyzed. The results apply to y as well. Let $P_x(t)$ be the covariance matrix of the localization errors of x at time t.

$$P_x(t) = \begin{bmatrix} \sigma_1^2(t) & \sigma_{12}(t) & \cdots & \sigma_{1n}(t) \\ \sigma_{21}(t) & \sigma_2^2(t) & & \sigma_{2n}(t) \\ \vdots & & \ddots & \\ \sigma_{n1}(t) & \sigma_{n2}(t) & & \sigma_n^2(t) \end{bmatrix} \quad (15)$$

where the components of $P_x(t)$ are defined by $\sigma_{ij}(t) = Cov\left[\tilde{x}_i(t), \tilde{x}_j(t)\right]$.

We would like to examine the evolution of P_x in time. For example, as shown in the previous section, when no correction mechanisms are applied, the covariance of the localization error of any two robots is zero i.e. for any $i \neq j$, $\sigma_{ij}(t) = 0$. And the components on the main diagonal of P_x grows linearly in time i.e. for any i, $\sigma_i^2(t) = t \cdot \sigma_0^2$. However, when error correction mechanisms are applied, the evolution of $P_x(t)$ is more complex.

Considering a group of robots performing EA, $P_x(t)$ can be derived from $P_x(t-1)$ in two stages. In the first - an error is added and in the second - meetings between robots are accounted for. The error addition stage is given by

$$P_x\left(t^-\right) = P_x\left(t-1\right) + I_M \cdot \sigma_0^2 \quad (16)$$

where I_M is the unit matrix of size M. To account for meetings, consider all pairs of robots which met at time t. Every meeting is considered separately[3]. Let r_i, r_j be two robots who have met at time t. The meeting process is described for robot r_i; r_j follows the same procedure in parallel. Upon meeting, r_i asks r_j "what is your estimation of my location?". r_j replies with

$$\begin{bmatrix} \hat{x}_j\left(t^-\right) \\ \hat{y}_j\left(t^-\right) \end{bmatrix} + \left(\begin{bmatrix} x_i\left(t\right) \\ y_i\left(t\right) \end{bmatrix} - \begin{bmatrix} x_j\left(t\right) \\ y_j\left(t\right) \end{bmatrix} \right) \quad (17)$$

Then, r_i sets his location estimation to the average of his previous estimation and the coordinates received from r_j, i.e.

$$\hat{x}_i\left(t\right) = \frac{\hat{x}_i\left(t^-\right) + \left(\hat{x}_j\left(t^-\right) + x_i\left(t\right) - x_j\left(t\right)\right)}{2} \quad (18)$$

$$\tilde{x}_i\left(t\right) = \frac{\tilde{x}_i\left(t^-\right) + \tilde{x}_j\left(t^-\right)}{2} \quad (19)$$

$$\sim N\left(0, \frac{\sigma_i^2\left(t^-\right) + \sigma_j^2\left(t^-\right) + 2\sigma_{ij}\left(t^-\right)}{4}\right) \quad (20)$$

To gain insight of Equation 20, observe two limit cases. In case $\sigma_i^2\left(t^-\right) = \sigma_j^2\left(t^-\right)$ and $\tilde{x}_i\left(t^-\right)$, $\tilde{x}_j\left(t^-\right)$ are independent (i.e. $\sigma_{ij}\left(t^-\right) = 0$), $\sigma_i^2\left(t\right) = \sigma_i^2\left(t^-\right)/2$ i.e. the variance was halved. In case $\sigma_i^2\left(t^-\right) = \sigma_j^2\left(t^-\right)$ and $\tilde{x}_i\left(t^-\right)$, $\tilde{x}_j\left(t^-\right)$ are fully correlated (i.e. $\sigma_{ij}\left(t^-\right) = \sigma_i^2\left(t^-\right)$), $\sigma_i^2\left(t\right) = \sigma_i^2\left(t^-\right)$ i.e. the localization was not improved. Note that after the update $\tilde{x}_i\left(t\right) = \tilde{x}_j\left(t\right)$ so $\tilde{x}_1\left(t\right)$ and $\tilde{x}_2\left(t\right)$ are fully correlated and another averaging of their location estimations will not reduce the error. To conclude, after a meeting between r_i and r_j

$$\begin{aligned} \sigma_i^2\left(t\right) &= \sigma_j^2\left(t\right) = \sigma_{ij}\left(t\right) \\ &= \frac{\sigma_i^2\left(t^-\right) + \sigma_j^2\left(t^-\right) + 2\sigma_{ij}\left(t^-\right)}{4} \end{aligned} \quad (21)$$

Note that other elements of P_x must be updated as well. Consider any robot r_k $(k \neq i, j)$,

$$\begin{aligned} \sigma_{ik}\left(t\right) &= E\left[\tilde{x}_i\left(t\right) \cdot \tilde{x}_k\left(t\right)\right] & (22) \\ &= E\left[\frac{\tilde{x}_i\left(t^-\right) + \tilde{x}_j\left(t^-\right)}{2} \cdot \tilde{x}_k\left(t^-\right)\right] & (23) \\ &= \frac{1}{2}\left(\sigma_{ik}\left(t^-\right) + \sigma_{jk}\left(t^-\right)\right) & (24) \end{aligned}$$

So $\sigma_{ik}\left(t\right)$ equals the average of $\sigma_{ik}\left(t^-\right)$ and $\sigma_{jk}\left(t^-\right)$. The process of updating P_x as a result of a meeting between r_i and r_j can be carried out by averaging rows i and j of P_x and afterward columns i and j. Note that the total sum of P_x does not change as a result of meetings so the average of P_x is given by

$$E\left[P_x\left(t\right)\right] = M \cdot \sigma_0^2 \cdot t/M^2 = \frac{\sigma_0^2}{M} \cdot t \quad (25)$$

So EA does not reduce the total amount of noise in P_x. Nevertheless, EA spreads the error from the main diagonal of P_x to the rest of the matrix. Since the robots' localization error is proportional only to the values of the main diagonal of P_x, spreading the error is desired.

The evolution of P_x, when the robots follow EA, can be described as follows. Initially: $P_x = 0$. Then, as the robots move, the values on the main diagonal of P_x starts growing. Due to meetings between robots, error from the main diagonal spread to the rest of P_x. Finally, a semi-steady state is achieved in which the rate of removing error from the main diagonal (almost) equals the rate of adding error to it. The analysis purpose is to predict the characteristics of such a semi-steady state. This is achieved by assuming that P_x comprises only two values: $\sigma_{diag}^2\left(t\right)$ and $\sigma_{cov}^2\left(t\right)$. It is assumed that for any i, $\sigma_i^2\left(t\right) = \sigma_{diag}^2\left(t\right)$ and for any $i \neq j$, $\sigma_{ij}\left(t\right) = \sigma_{cov}^2\left(t\right)$. The two latter assumptions are clearly false (P_x comprises many values). However, when considering a large group of homogeneous robots, it is a reasonable approximation to assume that the localization uncertainties related to most robots (the values of the main diagonal of P_x) are about the same. In the same spirit, it is reasonable to assume that the covariances between most robots are about the same.

By neglecting the main diagonal contribution to $E\left[P_x\right]$, $\sigma_{cov}^2\left(t\right)$ can be approximated by

$$\sigma_{cov}^2\left(t\right) \simeq E\left[P_x\left(t\right)\right] = \frac{\sigma_0^2}{M} \cdot t \quad (26)$$

[3] In case robot r have met two robots in the same time cycle. We assume that he first meets one of them and later - the other.

Consider any robot r_i. Let p_r be the probability that r_i meets another robot at time t (p_r was derived in section 3). In case r_i does not meet another robot at time t,

$$\sigma_i^2(t+1) = \sigma_i^2(t) + \sigma_0^2 \quad (27)$$

In case r_i meets r_j at time t,

$$\sigma_i^2(t+1) = \frac{\sigma_i^2(t) + \sigma_j^2(t) + 2\sigma_0^2 + 2\sigma_{ij}(t)}{4} \quad (28)$$

Using $\sigma_i^2 \simeq \sigma_j^2 \simeq \sigma_{diag}^2$, $\sigma_{ij} \simeq \sigma_{cov}$ and Equations 27 and 28 we get

$$\sigma_{diag}^2(t+1) = p_r \frac{\sigma_{diag}^2(t) + \sigma_0^2 + \sigma_{cov}^2(t)}{2} + (1-p_r)\left(\sigma_{diag}^2(t) + \sigma_0^2\right) \quad (29)$$

Equations 26 and 29 form a set of two difference equations. The fixed point solution for this set is given by

$$\sigma_{cov}^2(t) = \frac{\sigma_0^2}{M} \cdot t \quad (30)$$

$$\sigma_{diag}^2(t) = \left(\frac{2}{p_r}\frac{M-1}{M} - 1\right)\sigma_0^2 + \sigma_{cov}^2(t) \simeq \frac{2\sigma_0^2}{p_r} + \frac{\sigma_0^2}{M} \cdot t \quad (31)$$

So the components of P_x not on the main diagonal grows linearly in time at a rate of σ_0^2/M. The values on the main diagonal grow with the same rate. However, there is a constant gap of about $2\sigma_0^2/p_r$ between the values on main diagonal and the rest of the covariance matrix.

5 Using a Landmark

Consider a landmark placed in a fixed point in the environment. The robots know the exact coordinates of the landmark and every robot that is within the landmark sensing range updates his localization accordingly. As a consequence, every time a robot sense the landmark, it's localization error is zeroed. Furthermore, since his new localization is uncorrelated with the other robots, the covariance of the robot with all other robots is also reduced to zero. Note that any robot can be used as a landmark. It just needs to stay put and report his coordinates to every robot in his communication range.

In order to characterize the evolution of P_x, the same approximation as in the previous section will be used i.e. it is assumed that P_x comprises only two values: For any i, $\sigma_i^2(t) = \sigma_{diag}^2(t)$ and for any $i \neq j$, $\sigma_{ij}(t) = \sigma_{cov}^2(t)$. Fix any robot r_i. Let p_r be the probability that r_i meet another robot at time t and let p_l be the probability that r_i sense the landmark at time t. p_l is derived in Appendix A. First, an approximation for $E[P_x(t)]$ is presented. In case r_i sense the landmark at time t, one column and one row of P_x are zeroed i.e. a total sum of about $2M \cdot E[P_x(t)]$ is removed from P_x. Since there are M robots,

$$\sum P_x(t^-) = \sum P_x(t-1) + M \cdot \sigma_0^2 \quad (32)$$

$$E\left[\sum P_x(t)\right] = E\left[\begin{array}{l} \sum P_x(t^-) \\ -M \cdot p_l \cdot 2M \cdot E[P_x(t^-)] \end{array}\right] \quad (33)$$

using $E[\sum P_x(t)] = M^2 \cdot E[P_x(t)]$ we get the following difference equation,

$$E[P_x(t)] = (1-2p_l)\left(E[P_x(t-1)] + \frac{\sigma_0^2}{M}\right) \quad (34)$$

The fix point solution of Equation 34 is

$$E[P_x(t)] = \frac{1-2p_l}{2p_l}\frac{\sigma_0^2}{M} \simeq \frac{\sigma_0^2}{2Mp_l} \quad (35)$$

As before, $\sigma_{cov}^2(t)$ is approximated by $E[P_x(t)]$. In order to evaluate $\sigma_{diag}^2(t)$ observe three cases:

1. In case r_i does not meet another robot at time t,

$$\sigma_i^2(t+1) = \sigma_i^2(t) + \sigma_0^2 \quad (36)$$

2. In case r_i meets with robot r_j at time t,

$$\sigma_i^2(t+1) = \frac{\sigma_i^2(t) + \sigma_j^2(t) + 2\sigma_0^2 + 2\sigma_{ij}(t)}{4} \quad (37)$$

3. In case r_i sense the landmark at time t,

$$\sigma_i^2(t+1) = 0 \quad (38)$$

Since p_l is very small we assume that cases 2 and 3 are mutually exclusive. Using Equations 36-38 we get

$$\sigma_{diag}^2(t+1) = p_r \frac{\sigma_{diag}^2(t) + \sigma_0^2 + \sigma_{cov}^2(t)}{2} + (1 - p_r - p_l)\left(\sigma_{diag}^2(t) + \sigma_0^2\right) \quad (39)$$

The fix point solution of this difference equation is given by

$$\sigma_{diag}^2 = \frac{p_r \sigma_{cov}^2 + \sigma_0^2 (2 - p_r - 2p_l)}{p_r + 2p_l} \quad (40)$$

$$\simeq \frac{2\sigma_0^2}{p_r} + \sigma_{cov}^2 \quad (41)$$

To conclude,

$$\sigma_{cov}^2 \simeq \frac{\sigma_0^2}{2Mp_l} \quad (42)$$

$$\sigma_{diag}^2 \simeq \frac{2\sigma_0^2}{p_r} + \sigma_{cov}^2 \quad (43)$$

So a single landmark is sufficient to make the localization error bounded.

6 Discussion and Simulations

Simulations were used in order to validate the analytical results. In all figures, the solid lines are the theoretical estimations and the error bars (or markers) represent the simulation results. When presenting σ_{diag}^2, the average of the main diagonal of P_x is presented with its standard deviation. When σ_{cov}^2 is presented, the average of P_x without the main diagonal is presented.

In a scenario where EA is applied without a landmark, the localization error of a robot is given by

$$\tilde{x}(t) \sim \tilde{y}(t) \sim N\left(0, \sigma^2(t)\right) \quad (44)$$

$$\sigma^2(t) = \tfrac{2\sigma_0^2}{p_r} + \tfrac{\sigma_0^2}{M} \cdot t \quad (45)$$

i.e. there is a constant component and a time dependent component. When no error correction mechanisms are applied $\sigma^2(t) = \sigma_0^2 \cdot t$. Hence by applying EA, the error growth rate is reduced by a factor of M. However, a constant component is added. This constant component is a result of the time the odometry errors require to average over the robots. Nevertheless, even with EA, the error remains unbounded. The error can be bounded by introducing a landmark into the system. With a landmark, the localization error is given by Equation 44 with,

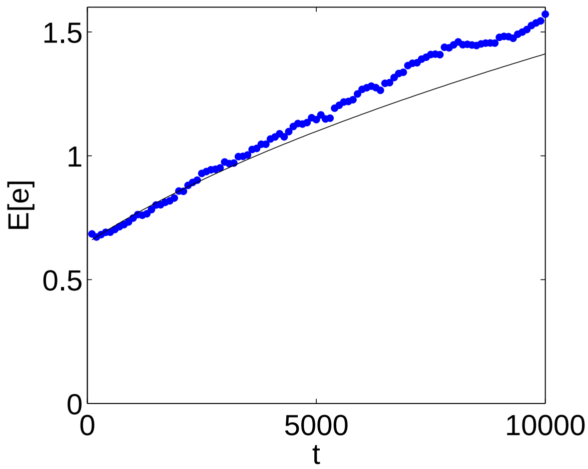


Figure 5: Average of the mean localization error over 50 runs.

$$\sigma^2(t) = \tfrac{2\sigma_0^2}{p_r} + \tfrac{\sigma_0^2}{2Mp_l} \quad (46)$$

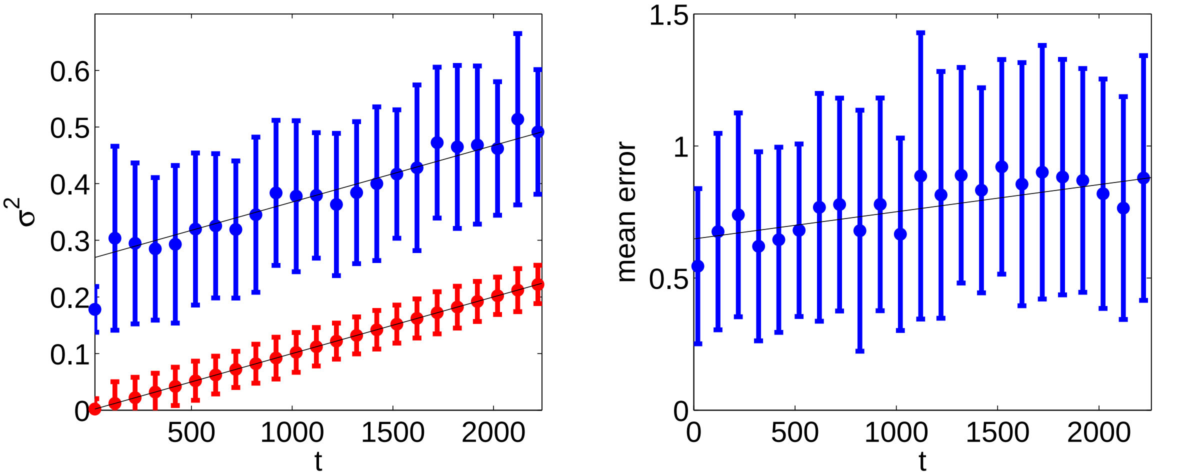
Recall that the expected localization error is given by,

$$E[e] = \sqrt{\frac{\pi}{2}\sigma^2(t)} \quad (47)$$

The values of σ_{cov}^2, σ_{diag}^2 and $E[e]$ for a single run are presented in Figure 4. The environment for this run was a torus of size 100×100, the group comprised $M = 100$ robots where $V = 3$, $v_0 = 1$ and $\sigma_0^2 = 0.01$. The experiments show that the estimations of σ_{diag}^2 and σ_{cov}^2 are very accurate. The mean error was found to be very noisy for a single run. Hence the average of the mean error over 50 runs is presented in Figure 5. The average is less noisy and it can be observed that the mean error is predicted well.

The values of σ_{cov}^2, σ_{diag}^2 for a single run on a torus with a landmark are presented in Figure 6a. The average of the mean error over 50 runs is presented in Figure 6b. The simulation results agree with the analysis.

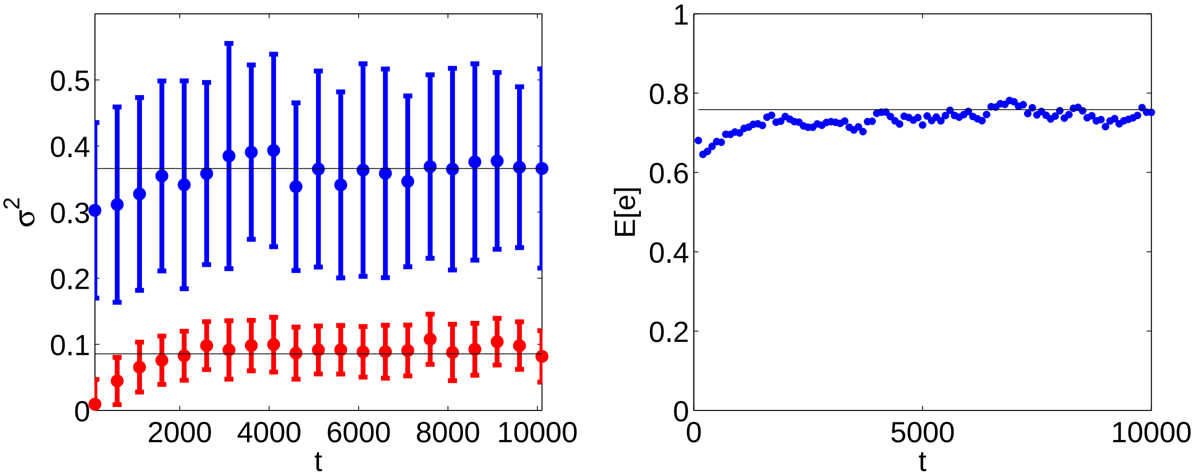
The simulations have shown that the approximations are accurate when the environment is a torus.



(a) The lower line is σ^2_{cov} and the upper is σ^2_{diag}.

(b) Mean error.

Figure 4: σ^2_{cov}, σ^2_{diag} and $E[e]$ for a single run on a torus where $M = 100$, $A = 100^2$, $V = 3$, $v_0 = 1$ and $\sigma_0^2 = 0.01$. The solid lines are the theoretical estimations and the error bars are the simulation results.



(a) The lower line is σ^2_{cov} and the upper is σ^2_{diag}.

(b) Average of the mean error over 50 runs.

Figure 6: (6a) σ^2_{cov}, σ^2_{diag} for a single run and (6b) $E[e]$ averaged over 50 runs. The environment was a torus with a single landmark. Parameters: $M = 100$, $A = 100^2$, $V = 3$, $v_0 = 1$ and $\sigma_0^2 = 0.01$. The solid lines are the theoretical estimations and the error bars are the simulation results.

The values of σ^2_{cov} and σ^2_{diag} for a single run in a "9-rooms" environment (see Figure 7a) are presented in Figure 7b. For a 9-rooms environment, σ^2_{cov} and the growth rate of σ^2_{diag} are predicted well but the constant gap between σ^2_{cov} and σ^2_{diag} is higher then expected. Recall that this gap is the result of the time required to average the error over the robots and is given by $2\sigma_0^2/p_r$. On the torus, the robots travel freely hence at every time step there is a probability of p_r to meet a "fresh" robot i.e. a robot such as the localization covariance with it is σ^2_{cov}. On the contrary, in the 9-rooms environment, there is a high probability to meet a "dirty" robot i.e. a robot with high shared covariance due to a recent meeting. Meeting a "fresh" robot reduces the localization error much more efficiently that meeting a "dirty" one. Putting it another way, for the 9-rooms environment the effective p_r is lower than given in Section 3 hence the gap is larger. Simulations results for more environments can be found in Figure 8. The results are the same as for the 9-rooms environment.

7 Previous Work

About ten years ago, Sanderson[5, 3] have proposed a cooperative localization mechanism almost similar to EA. In his work, an optimal Kalman filter (KF) is used in the localization process i.e. when two robots exchange location information, they perform a weighted average based on their localization quality. The robot with the more precise localization (lower σ value) is given a higher weight. In order to calculate the weights, the meeting robots need to know their covariance. Hence, Sanderson have proposed a central (non-distributed) correction mechanism. He have also presented a distributed algorithm for the fully symmetric case: homogeneous group and a complete relative position measurement graph (RPMG) i.e. at every time step all robots meet all robots. In EA, a suboptimal kalman filter is applied i.e. the weights of both robots are equal. Hence the robots are not required to maintatin the covariance matrix.

Roumeliotis and colleagues[6, 7, 8, 9, 10] have presented a distributed version of KF in which the computation required to maintain the covariance matrix is distributed between the robots. However, every meeting between two robots implies an update of $2M$ components of the covariance matrix. Furthermore, all robots must be aware of every update of the covariance matrix. In the distributed KF of Roumeliotis *et al.*, every meeting implies a computation complexity of $\Theta\left(M^2\right)$ and communication between all robots, so their algorithm does not scale well. In order solve that problem, Mourikis and Roumeliotis have proposed to reduce the computation and communicating loads by lowering the frequency of relative observations[11]. Martinelli have proposed to use hierarchical structure of Kalman filters[12] i.e. the robots are divided into groups, relative observations and corrections are performed within each group, inter-group corrections are performed only between the group leaders hence reducing the computation and communication complexity. On the contrary, in EA, the computation complexity implied by a meeting is $\Theta(1)$ and the only communication required is between the two meeting robots.

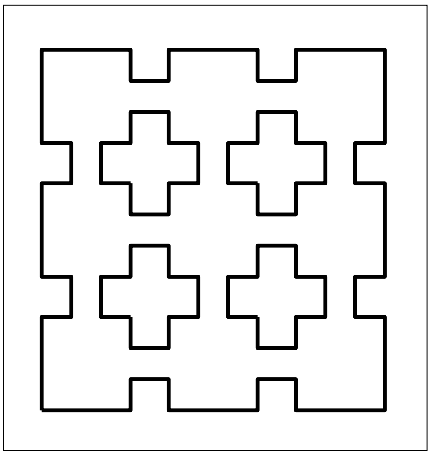
In order to compare between KF and EA, a centralized version of KF was implemented. In KF, when two robots r_i, r_j meet, r_i updates his localization to be

$$\hat{x}_i(t) = \alpha\hat{x}_i\left(t^-\right) + (1-\alpha)\left(\hat{x}_j\left(t^-\right) + x_i(t) - x_j(t)\right) \quad (48)$$

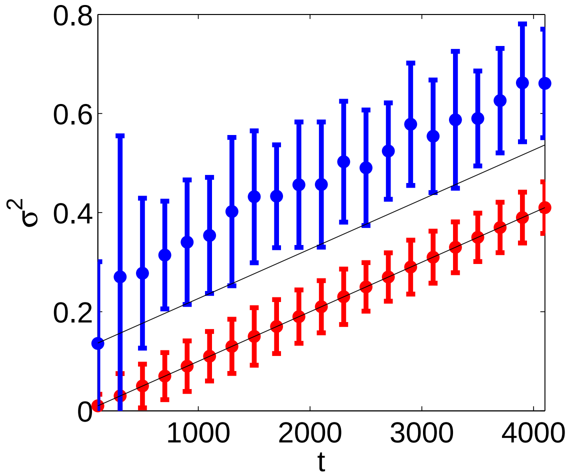
$$\alpha = \frac{\sigma_j^2\left(t^-\right) - \sigma_{ij}\left(t^-\right)}{\sigma_i^2\left(t^-\right) + \sigma_j^2\left(t^-\right) - 2\sigma_{ij}\left(t^-\right)} \quad (49)$$

where α is chosen such as to minimize $\tilde{x}_i(t)$ (compare with Equation 18 in which $\alpha = \frac{1}{2}$).

Observe Figure 9 for experimental comparison between KF and EA. Result on a torus without a landmark are presented in Figure 9a. At the begining of the process, until $t = 500$, KF performs better then EA due to its faster error dispersion rate. However, KF causes the covariance matrix to include more energy then EA (observe the two bottom lines in the figure). Hence, KF reduce the constant component of the localization error but increase the slope of the time dependent component. So, in time, EA outperforms KF. This can be explained as follows. Consider a meeting between two robots r_i and r_j. Assume that the localization of r_i is better then the localization of r_j i.e. $\sigma_i^2 < \sigma_j^2$. So when applying KF, the weight given to r_i is higher. The robots are homogeneous, so $\sigma_i^2 < \sigma_j^2$ implies that r_i have met more robots prior to his meeting with r_j. Hence r_i shares more covariance with

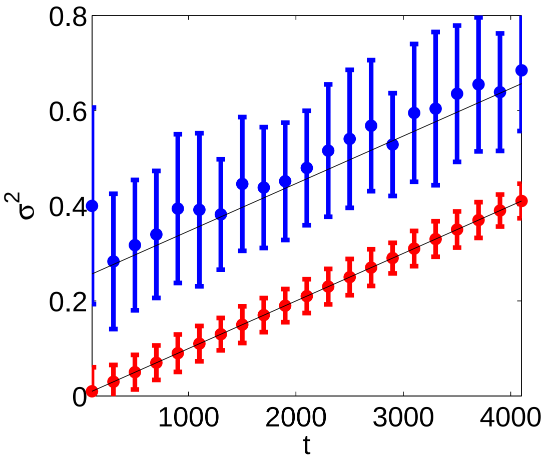


(a) 9-rooms environment.

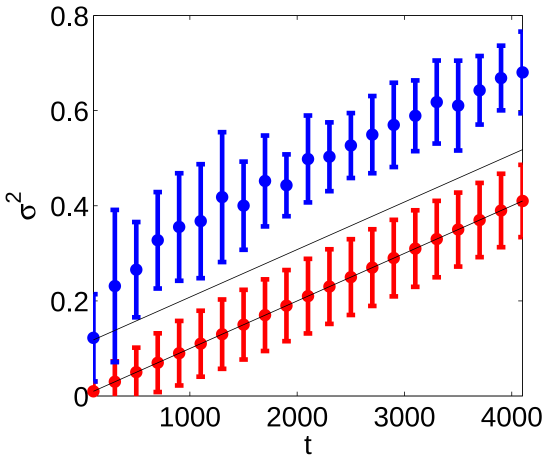


(b) σ^2_{cov} and σ^2_{diag} for a single run in 9-rooms where $M = 100$, $V = 3$, $v_0 = 1$ and $\sigma_0^2 = 0.01$. The lower line is σ^2_{cov} and the upper is σ^2_{diag}.

Figure 7: Experiment on a 9-rooms environment.



(a) Box environment.



(b) Ring environment.

Figure 8: σ^2_{cov} and σ^2_{diag} for a single run where $M = 100$, $V = 3$, $v_0 = 1$ and $\sigma_0^2 = 0.01$. The lower line is σ^2_{cov} and the upper is σ^2_{diag}.

other robots. Upon meeting, these high covariances will be copied to r_j due to r_i's high weight thus contributing to the overall value of the covariance matrix.

Experimental comparison on a torus with a landmark are presented in Figure 9b. When the environment include a landmark, KF outperforms EA due to its faster error dispersing rate. KF is able to transfer the quality localization from the landmark to the robots more efficiently. Since meetings with the landmark continuously remove energy from the covariance matrix, the bad side effect of KF is negligible.

Roumeliotis and Rekleitis were the first to analyze the performance of KF[13, 14]. They have considered homogeneous robots with complete RPMG. Later, Mourikis and Roumeliotis have extended the analysis to include heterogeneous groups and general RPMG[15, 10]. Mourikis and Roumeliotis have analyzed KF assuming a fixed RPMG i.e. every robot average its location with a fixed set of other robots[4]. By fixing the RPMG, they have been able to obtain an exact analysis of the localization process. In this work, the RPMG is not fixed hence approximations must be used.

The model used by Mourikis and Roumeliotis is more suitable to ground robots then IEM. In their model, every robot sense its orientation using a compass and updates his localization based on the distance and direction traveled. The localization errors result from the wheel encoders and compass noises. So the localization errors added at each time step are independent but are effected by the robot state (heading, speed). We intend to apply our formalizm on more realistic models like the one considered in [15] in the future. Even though the model of Mourikis and Roumeliotis is more general then IEM, the analysis of both models produce similar results. The main similarities are:

- The error comprises a time dependent term and a constant term. The time dependent term is monotonically increasing (in time) and is dependent solely on the number of robots and the quality of the odometry. In particular, it is not dependent on the RPMG. Observe Equation 31. The time dependent part is dependent of σ_0^2 (odometry noise) and M but is independent of p_r (a characteristic of the RPMG).
- When a single robot (or more) have access to absolute position measurement, the error of all robots become bounded. In our work, this happens when a landmark is introduced.

[4]They have also considered changes of the RPMG but their results discuss the the system state after stabilization.

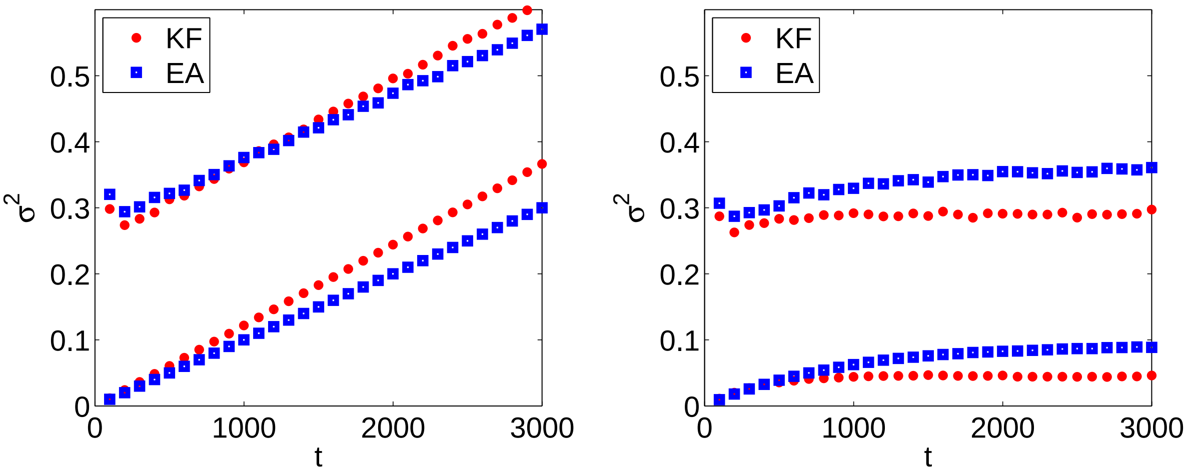
With resemblance to KF, Fox *et al.* have proposed to average the location estimations between robots[16]. In their work, every robot estimate its location using Monte Carlo localization[17, 18] i.e. every robot maintains a cloud of points in space with a probability attached to every point. The robot location estimation is the probability function implied by the cloud. When two robots sense each other, their clouds are averaged.

Kurazume and colleagues[19, 20, 21] have proposed a strategy based on “portable landmarks”. In this scheme, every time a robot moves, other robots are holding still while following the robot movement with their sensors. The viewing robots supply the moving one with a localization better then given by his own odometry. Later, several other works were carried out using this scheme, see [22, 23]. Since this strategy is not solely odometry based, it is more resilient to correlation between odometry errors. On the downside, when applying this scheme, the robots' movements are limited. Where in EA and KF, no special movement pattern is required, the robots are free to go wherever the task they perform requires.

8 Conclusion

We have presented the error averaging (EA) localization scheme inspired by the optimal Kalman filter (KF) proposed by Sanderson[3] and Roumeliotis *et al.*[9]. The idea behind EA is simple: Whenever two robots meet, they average their location estimations. EA requires considerably less communication and computation then KF. Furthermore, EA produce better results when no absolute localization information is available to the robots.

In case the robots have no access to absolute localization information, EA's localization error is composed of two components. A constant component and a monotonically increasing time dependent component. The constant component result



(a) Torus without a landmark.   (b) Torus with a landmark.

Figure 9: Comparison between EA (blue squares) and KF (red dots) under IEM. The bottom lines correspond to the mean value of the covariance matrix ($E[P_x]$) and the upper - the mean square error ($E[|e|]$). All values are averaged over 50 runs where $M = 100$, $V = 3$, $v_0 = 1$ and $\sigma_0^2 = 0.01$.

from the time the error require to propagate between the robots and is a function of the odometry quality and the frequency of meetings. The time dependent component results from the error accumulated by the robots and is a function of the odometry quality and the number of robots i.e. it is independent of the frequency of meetings. In case some robots have access to absolute localization (e.g. a landmark), the localization errors of all robots become bounded.

Simulations showed that the analysis is accurate on the torus. When the environment includes obstacles, the time dependent component is predicted well but the constant is not. This is because p_r, the probability of meeting, was derived for the torus. When the environment comprises obstacles, the effective p_r is lower then calculated hence the constant component is larger.

In this paper we have used a soft "thermodynamic" analysis i.e. relaxing assumptions were used. The analysis goal was not to derive hard lower or upper bounds but rather to characterize the robots expected behavior. In particular, to predict the expected localization error. We believe that this kind of soft analysis can be beneficial in many other cases.

References

[1] Johann Borenstein, H. R. Everett, and Liqiang Feng. *Navigating Mobile Robots: Systems and Techniques*. A. K. Peters, Ltd., Natick, MA, USA, 1996.

[2] Sebastian Thrun. Robotic mapping: a survey. In *Exploring Artificial Intelligence in the New Millenium*, pages 1–35. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2003.

[3] Arthur C Sanderson. A distributed algorithm for cooperative navigation among multiple mobile robots. *Advanced Robotics*, 12:335–349(15), 1997.

[4] Eric W. Weisstein. Circle-circle intersection. from mathworld–a wolfram web resource.

[5] Arthur C Sanderson. Cooperative navigation among multiple mobile robots. In *Proc. 3rd Int Symp on Distributed Autonomous Robotic Systems.*, pages 1351 – 1357, Saitama, Japan., 1996.

[6] S.I. Roumeliotis. Distributed multirobot localization. Technical report, Calif. Inst. Technol., Pasadena, 2002.

[7] S.I. Roumeliotis and G.A. Bekey. Collective localization: a distributed kalman filter approach to localization of groups of mobile robots. In *PROC IEEE INT CONF ROB AUTOM.*, volume 3, pages 2958–2965, 2000.

[8] S. I. Roumeliotis. *Robust mobile robot localization: from single-robot uncertainties to multirobot interdependencies.* PhD thesis, Elec. Eng. Dept. Univ. South. Calif., Los Angeles, CA, 2000.

[9] S.I. Roumeliotis and G.A. Bekey. Distributed multirobot localization. *Robotics and Automation, IEEE Transactions on*, 18(5):781–795, Oct 2002.

[10] Anastasios I. Mourikis and Stergios I. Roumeliotis. Predicting the performance of cooperative simultaneous localization and mapping (c-slam). *The International Journal of Robotics Research*, 25(12):1273–1286, 2006.

[11] A.I. Mourikis and S.I. Roumeliotis. Optimal sensing strategies for mobile robot formations: Resource-constrained localization. In *Robotics: Science and Systems*, Cambridge, Massachusetts, USA, June 2005. Massachusetts Institute of Technology.

[12] Agostino Martinelli. Improving the precision on multi robot localization by using a series of filters hierarchically distributed. In *Proc. IEEE/RSJ International Conference on Inteligent Robots and Systems*, San Diego, CA, USA, Oct 2007.

[13] Stergios I. Roumeliotis and Ioannis M. Rekleitis. Analysis of multirobot localization uncertainty propagation. In *Proc. IEEE/RSJ Int. Conf. Robot. Intell. Syst.*, pages 1763 – 1770, Las Vegas, NV, Oct. 2003.

[14] Stergios I. Roumeliotis and Ioannis M. Rekleitis. Propagation of uncertainty in cooperative multirobot localization: Analysis and experimental results. *Auton. Robots*, 17(1):41 – 54, Jul 2004.

[15] A.I. Mourikis and S.I. Roumeliotis. Performance analysis of multirobot cooperative localization. *Robotics, IEEE Transactions on*, 22(4):666–681, Aug. 2006.

[16] Dieter Fox, Wolfram Burgard, Hannes Kruppa, and Sebastian Thrun. A probabilistic approach to collaborative multi-robot localization. *Autonomous Robots*, 8(3):325–344, June 2000.

[17] Dieter Fox, Sebastian Thrun, Wolfram Burgard, and Frank Dellaert. Particle filters for mobile robot localization. In *Sequential Monte Carlo Methods in Practice.* Springer-Verlag, 2001.

[18] Sebastian Thrun, Dieter Fox, Wolfram Burgard, and Frank Dellaert. Robust monte carlo localization for mobile robots. *Artificial Intelligence*, 128(1-2):99 – 141, 2001.

[19] R. Kurazume, S. Nagata, and S. Hirose. Cooperative positioning with multiple robots. In *IEEE International Conference on Robotics and Automation*, pages 1250–1257 vol.2, May 1994.

[20] R. Kurazume, S. Hirose, S. Nagata, and N. Sashida. Study on cooperative positioning system (basic principle and measurement experiment). In *Robotics and Automation, 1996. Proceedings., 1996 IEEE International Conference on*, volume 2, pages 1421–1426 vol.2, Apr 1996.

[21] Ryo Kurazume and Shigeo Hirose. An experimental study of a cooperative positioning system. *Autonomous Robots*, 8(1):43–52, January 2000.

[22] Ioannis Rekleitis, Gregory Dudek, and Evangelos Milios. Multi-robot exploration of an unknown environment, efficiently reducing the odometry error. In *International Joint Conference in Artificial Intelligence*, volume 2, pages 1340–1345, Nagoya, Japan, August 1997. Morgan Kaufmann Publishers, Inc.

[23] Ioannis Rekleitis, Gregory Dudek, and Evangelos Milios. Multi-robot collaboration for robust exploration. *Annals of Mathematics and Artificial Intelligence*, 31(1):7–40, October 2001.

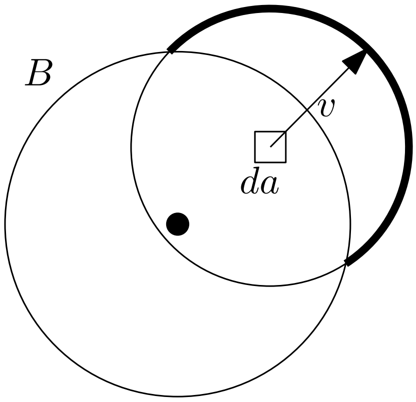


Figure 10

A Deriving p_l

Consider any robot r. In this section we derive p_l i.e. the probability of r sensing the landmark at time t. Let B be a circle of radius V around the landmark, see Figure 10. Let $r(t)$ be the location of robot r at time t. r collides with the landmark at time t if $r(t) \in B$ and $r(t-1) \neq B$ (see the derivation of p_r in Section 3).

To derive the p_l, consider an infinitesimal area of size $da \subseteq B$,

$$\begin{aligned} p_l &= Pr\left[r(t) \in B \wedge r(t-1) \notin B\right] && (50) \\ &= \int_B Pr\left[r(t) \in da \wedge r(t-1) \notin B\right] da && (51) \\ &= \int_B \begin{array}{l} Pr\left[r(t) \in da\right] \\ \cdot Pr\left[r(t-1) \notin B \,|\, r(t) \in da\right] \end{array} da && (52) \end{aligned}$$

Assuming r has the same probability to be anywhere in the area we can write

$$Pr\left[r(t) \in da\right] = da/A$$

In case $r(t) \in da$, we assume that $r(t-1)$ is uniformly spread over the circle of radius v_0 (the step length) with a center at $r(t)$. So $Pr\left[r(t-1) \notin B \,|\, r(t) \in da\right]$ equals the part of this circle which is not inside B (the bold arc in Figure 10). Let $r(t) = (x_r(t), y_r(t))$ and WLOG assume that the landmark location is $(0,0)$. We face the problem of calculating the intersection of two circles at distance $d = \sqrt{x_{r'}^2 + y_{r'}^2}$. ϕ is given by (see [4]):

$$\phi = \begin{cases} 2\pi - 2\operatorname{asin}\left(\frac{\sqrt{4d^2V^2 - (d^2 - v^2 + V^2)^2}}{2dv}\right) & \text{if } d \geq V - v \\ 0 & \text{else} \end{cases} \quad (53)$$

so

$$Pr\left[r(t-1) \notin B \,|\, r(t) \in da\right] = \frac{2\pi - \phi}{2\pi} \quad (54)$$

and finally

$$p_l = \int_B \frac{2\pi - \phi}{2\pi A} da \quad (55)$$

p_l can be derived numerically.